The Abbaco mathematical culture (1300 - 1500)

Albrecht Heeffer
Ghent University, Belgium

Mathematical Cultures I

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Overview

• Basic questions on Mathematical Cultures
• A framework from mathematics education
• The abbaco tradition
• Relevance for a contemporary understanding of mathematical culture
Basic questions

• What constitutes a mathematical culture?
• Mathematical cultures and society?
  – How does it emerge?
  – What is its role in society?
  – How is it supported by society?
• How do different mathematical cultures relate to each other?
Framework

• Alan Bishop, *Mathematical enculturation*, 1988
  – Mathematics is not a set of abstract truths independent from society and culture
  – Mathematics is a cultural product
  – Mathematics can have different interpretations, values, uses in different cultures (ethno-mathematics)
  – Mathematical culture is produced and maintained by (formal) enculturation

• “a creative, interactive process engaging those living the culture with those born into it, which results in ideas, norms, and values which are similar from one generation to the next but inevitably must be different in some way due to the recreation role of the next generation” p.88
Framework

• 6 universal activities of mathematical cultures
  – Counting
  – Locating (spatial notions)
  – Measuring
  – Designing
  – Playing
  – Explaining (classification)
Abbaco tradition

• Northern Italy: 1300-1500
• More general: introduction and use of Hindu-Arabic numerals in Europe by lay culture
• A mathematical culture *par excellence*
  – strong in formal enculturation
  – representing all 6 universal activities
  – supporting the mercantile society (sedentary merchant economy, banks, bookkeeping)
Abbaco tradition

activities of mathematical cultures
• Counting
  – Hand reckoning
  – Numeration with Hindu-Arabic numerals
  – librettini, algorithms (integers, fractions, surds)
• Locating (spatial notions)
  – Practical geometry, instruments,
• Measuring
  – Surveying, measuring plane figures and volumes (gauging)
• Designing
  – Perspective, fortification, machines
• Playing
  – Recreational problems, chess, conjuring
• Explaining (classification)
  – Tables, calenders, mnemonic devices
Two traditions of mathematical practice

Scholarly traditions
- scholastic in nature
  - ‘official’ texts and commentaries
- learned languages
  - Akkadian, Greek, Latin
- protective and conservative
- supported by authorities
  - emperor, church, patrons
- well-studied in history of science

Subscientific traditions
- oral tradition
- vernacular
- master-apprentice relations
- open to foreign influences
- supported by lay culture
  - merchants, surveyors, craftsmen, artisans
- underrepresented in history of science
Two traditions of arithmetic in medieval Europe

<table>
<thead>
<tr>
<th>Latin scholarly tradition</th>
<th>Sub-scientific tradition</th>
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<tr>
<td>• Boethius: 6th century</td>
<td>• Fibonacci: 1228</td>
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<td>• Gerbert: c. 996</td>
<td>• Jacopo da Firenze: 1307</td>
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<td>• Bernelinus: 11th century</td>
<td>• Paolo Gherardi: 1328</td>
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<td>• Johannes Hispalensis: c.1150</td>
<td>• Paolo dell’abbaco: 1339</td>
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<td>• Alexander of Villedieu: 1200</td>
<td>• Dardi of Pisa: 1344</td>
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<td>• Turchillus De Ardena: c.1200</td>
<td>• Giovanni de Danti: 1370</td>
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<tr>
<td>• Fibonacci: 1202, 1228</td>
<td>• Gilio of Siena: 1374</td>
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<td>• Sacrobosco: c. 1220</td>
<td>• Antonio de’ Mazzinghi: 1380</td>
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<td>• Jordanus de Nemore: c.1240</td>
<td>• Piero della Francesca: 1465</td>
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<td>• Jean de Murs: 1343</td>
<td>• Maestro Benedetti: 1470</td>
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Historiography

• Pietro Cossali. *Origine, trasporto in Italia, primi progressi in essa dell'algebra* (1797-9)
  – First history on algebra to deal with the abbaco tradition

  – Mathematician, professor at the Sorbonne
  – Book and manuscript collector
  – First transcriptions of abbaco manuscripts
  – Pierro della Francesca
Historiography

- Baldassarre Boncompagni (1868-1887)
  - *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche*
  - Book collector
  - Publication of transcriptions
- Louis Karpinski (1929)
- Gino Arrighi (1964-1987)
- Kurt Vogel (1977)
- Rafaella Franci (1982-)
- Elisabetta Ulivi (2002-)
- Jens Høyrup (2007)
Some published manuscripts

- c.1290, Columbia X511 A13 (Vogel 1977)
- 1307, Jacopa di Firenze (Høyrup 2007)
- 1310, *Liber habaci* (Florence, BNC, Magl.XI, 88)
- 1339, Paolo dell’abaco, *Trattato d’Aritmetica* (Arrighi 1964)
- c.1340, Maestro Biagio, *Chasi exenplari* (Pieraccini 1983)
- c.1344, Maestro Dardi, *Aliębraa argibra* (Franci 2001)
- c.1380, M° di Mazzinghi *Trattato di Fioretti* (Arrighi 1967)
- c.1390, Maestro Gilio, Questioni d’algebra (Franci, 1983)
- c.1417, Anonymous, Arte giamata aresmetica (Rivolo, 1983)
- c.1440, Giovanni di Bartolo, *Certi chasi* (Pancanti 1982)
- c.1460, Pier Maria Calandri, *Tractato d’abbacho* (Arrighi, 1974)
- c.1470, Maestro Benedetti (Salomone 1982)
Abbaco tradition

The common story: derived from
• al-Khwārizmī’s (Arithmetic and Algebra)
  – Latin translations
    • Arithmetic, Latin algorisms, DA,
    • Algebra: Robert of Chester (c. 1145), Gerard of Cremona (c. 1150), Guglielmo de Lunis (c. 1250)
    • Geometry: Liber embadorum (Plato of Tivoli)
• Fibonacci
  – Arithmetic and algebra: Liber Abbaci (1202)
  – Geometry: De practica geometrie (c. 1220)
Abbaco tradition

The great book syndrome exposed

- Not a consequence of Fibonacci’s *Liber abbaci* as posed by Van Egmond, Ulivi, ...
- Emerged not in Italy but in Provençe-Catalan region in the 12th century (before Fibonacci!)
- (Fibonacci mentions Provençe)

Characteristics

- Arabic, Chinese, Indian and Byzantian influences
- Vernacular (mostly Italian)
- Coherent practice during the 14th and 15th century
- About 200 extant manuscripts (van Egmond 1980)
Abbaco tradition

Arguments for a common tradition (Høyrup):

• Chapter 9: *Hic incipit magister castellanus*
  – (not in Boncompagni’s edition, see Høyrup, 2010)

• abbaco algebra:
  – not depending on al-Khwārizmī or Latin translations
    • rules for non-normalized equations
    • no arabisms or latinisms
    • abbaco symbolism close to Mahreb practices (ghubār)
  – closer to the Arabic *muʿāmalāt* tradition
Social context

- *Maestri d’abbaco* teaching in bottega’s in mercantile cities of northern Italy
- Family relations (Calandri’s: 5 generations)
- Socio-economic status: middle-class
- Students were sons of merchants and artisans,
- Some famous: Dante, Leonardo da Vinci
- Employed by the city or private
- Books collected by rich merchants and bankers
  - e.g. Strozzi family
  - well-preserved
  - some are nicely illustrated
Scuolo d’abbaco

• Ulivi (2002):
  – Earliest record in Italy 1284
  – 20 scuolo in Florence active between 1340 and 1510
• About 1200 students in Florence in 1343
• Students were boys aged 10-14
• Subjects: 7 mute
  1) numeration, addition, subtraction and the librettine, or tables of multiplication,
  2) to 4) on division with increasing complexity
  5) operations on fractions,
  6) the rule of three with business applications and
  7) the monetary system and problems of exchange
• The abbaco books were NOT used by the students, but to teach other abbaco masters
Typical abbaco book

• 200-600 pages
• Nicely illustrated
• Divided in chapters
  – Religious invocations
  – Numeration and operations
  – Multiplication tables
  – Hand reckoning
  – Tarifs and tables
  – Calenders and lunar tables
• Problems and solutions

Paolo dell’abbaco (Plimpton 195)
Hand reckoning
Business problems

• Rule of three
• Exchange of money (cambio)
• Barter (barrato)
• Partnership and partnership in time
• Interest and discount (meritare e scontare)
• Repayment of loans (recare a termini)
• Alligation
Classic algebraic problems

• Divide a number into two parts …
• Find me a number such that …
• Numbers in proportion
• Numbers in continuous proportion
• Indeterminate problems (100 fowls)
• Chinese remainder theorem
Recreational problems

- Linear problem in several unknowns
  - Men find a purse (Mahāvīra c. 850)
  - Men buy a horse
  - Men have some money
- Cistern problems (shared work)
- Business trips (earn and spend)
- Legacy problems
Geometrical problems

- Measurement
  - of plane figures
  - of solids
  - distances
- Gauging
- Building

Practical Geometry (Plimpton 167)
Rigid rhetorical structure

1. Problem enunciation
2. Choice of the rhetorical unknown
3. Manipulation of polynomials
4. Construction of an ‘equation’ solvable by a standard rule
5. Root extraction
6. Numerical test
Example 1

Earliest known abacus manuscript on algebra

Jacopo da Firenze, ms. Vat. Lat. 4862, f. 39v (1307)
1. Enunciation of the problem

- Someone makes two business trips. On the first he makes a profit of 12. On the second he wins in the same proportion and when he ends his trip he found himself with 54. **I want to know** with how much he started with.

2. Choice of the unknown

Uses a (modified of combined) unknown quantity of the problem as the rhetorical unknown

• Pose that one begins with one cosa.

• Poni che se movesse con una cosa.
3. Manipulating polynomials

Nel primo viaggio guadangniò 12, dunque, compiuto il primo viaggio si truova 1 cosa e 12, adunque manifestamente apare che d’ongni una cosa faegli 1 cosa e 12 nel primo viaggio. Adunque, se ogni una cosa fae una cosa e 12, quanto far`a una cosa e 12. Convienti multiprichare una cosa e 12 via una cosa e 12 e partire in una cosa. [f. 30v]. Una cosa e 12 via una cosa e 12 fanno uno cienso e 24 cose e 144 numeri, il quale si vuole partire per una cosa e deve venire 54. E perci`o multipricha 54 via una cosa, fanno 54 cose, le quali s’aguagliano a uno cieno e 24 cose e 144 numeri. Ristora ciaschuna parte, cio[è] di chavare 24 cose di ciaschuna parte.
Equations?

• And on the first trip he wins 12. Then completing his first trip he finds 1 cosa and 12.
• It is then also manifest that for each cosa one obtains 1 cosa and 12 on the first trip. How much does this become in the same proportion after the second trip?
• It is appropriate to multiply one cosa and 12 with one cosa and 12 which makes one censo and 24 cosa and 144 numbers, which will become 54.
• And therefore multiply 54 with one cosa. Makes 54 cose, which is equal with one censo and 24 cose and 144 numbers.
• Restore each part, therefore subtract 24 cose from each part.

\[
x + 12
\]

\[
x + 12 \div \frac{54}{x + 12}
\]

\[
(x + 12)(x + 12)
\]

\[
x^2 + 24x + 144
\]

54x

\[
x^2 + 24x + 144 = 54x
\]

\[
x^2 + 144 = 30x
\]
4. Constructing an ‘equation’

• You will have that 30 cose are equal to one censo and 144 numbers.

• Averai che 30 cose sono iguali a uno cienso e 144 numeri.

\[ 30x = x^2 + 144 \]

Arabic type V ‘equation’
5. Root extraction

Applying a cannonical recipe:

- Divide in one censo, which becomes itself. Then take half of the cose, which is 15. Multiply by itself which makes 225, subtract the numbers which are 144, leaves 81. Find its [square] root which is 9. Subtract it from half of the cose, which is 15. Leaves 6, and so much is the value of the cosa.


\[
bx = ax^2 + c \\
bx = x^2 + c \\
x = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c} \\
x = \frac{30}{2} - \sqrt{\left(\frac{30}{2}\right)^2 - 144} = 6
\]
6. The test

- And if you want to prove this, do as such. You say that on the first trip one wins 12 and with the 6 one started with, one has 18. So that on the first trip one finds 18. Therefore say as such, of every 6 I make 18; what makes 18 in the same proportion? Multiply 18 with 18, makes 324. Divide by 6, this becomes 54, and it is good.

Et se la voi provare, fa così. Tu di’ che nel primo viaggio guadagnio 12 et con 6 se mosse a 18. Siché nel primo viaggio se trovò 18. E però di’ così, se de 6 io fo 18, que farò de 18 a quella medesema ragione? Multipricha 18 via 18. Fa 324. Parti in 6, che ne vene 54, et sta bene.
Contributions by abbacists

- Education of the merchant class in reckoning with Hindu-Arabic numerals
- Establishment of a practice of problem solving with many applications
- Expansion of the number concept
- The second unknown
- Preparation for a solution to the cubic
- Creating the concept of objective ‘value’
- First recognized use of symbolism
- New mathematical techniques: e.g. induction
Creation of ‘value’

• Very chaotic and uncertain circumstances
  – Monetary units, units of measure, gold contents of coins differ between cities
  – No standards for determining time, length, volume

• Mercantile system is based on reciprocal relations
  – trust, fairness, objectivity
  – no exchange without ‘objective value’
  – Foucault (*Les mots et les choses*, 1966)

• Double-entry bookkeeping coincides with algebra

• Always ‘exact’ values in abbaco algebra
  – no approximations, acceptance of irrational quantities
Beginning of symbolism

- Beginning of the fifteenth century
- Existing practice of scratchpad calculations
- First recognized as a method different from rhetorical algebra in a family of manuscripts
- BNCF, Magl. Cl. XI. 119
Beginning of symbolism


• The abbaco master solves all problems in 2 ways:
  – Rethorically (*per scritura*)
  – Symbolically (*figuratamente*)
  – “I showed this symbolically as you can understand from the above, not to make things harder but rather for you to understand it better. I intend to give it to you by means of writing as you will see soon”.

• Early symbolism had an epistemic function just as other non-discursive elements in abbaco texts (justification schemes for operations)
From tables to induction

Analytic
• memorization
  – librettini
  – eastern day tables
• calculations
  – compound interest
  – chord tables
  – gauging rods and tables
  – moon cycles
  – eastern date calculation
• number-theoretical properties
  – progressions
  – chessboard problem
  – remainder problems

Synthetic
• units of weight and measure
• monetary units and exchange
• fineness of gold coins
• tariffs and taxes
• astronomical tables
Abbaco tables

- Chord tables
Abbaco tables

- Compound interest tables
- Ottob. lat. 3307 f. 225-233r

parameters:
  - 20 years
  - 6 to 20% in 0.5% increments
  - cumulated interest in monetary units (unit, 20th, 12th)
Abbaco tables: chessboard

- sum of a geometric progression
- probably of Indian origin
- many Arabic sources
- Fibonacci’s *Liber Abbaci*
  - repeated squaring to get $2^{64} - 1$
- also in Ottob. lat. 3307 f. 24r (c. 1465)
Abbaco tables

- Arithmetical progressions and squares
- Ottob. lat. 3307 f. 25r (c. 1465)
Abbaco remainder tables

Ms. BNC Florence

- Conv.Soppr.G7.1137
- c.1395 (van Egmond)
- Unpublished
- f. 421r-421v
- “Questo si chiama la tavola del numero infinito cioè è sanza fine”
Abbaco remainder tables

Other manuscripts (unpublished)

“questa reghola non arebbe mai fine e chiamasi la reghola del numero sanza fine”

- α (c.1417), ff. 141
- Florence, Magl.Cl.XI.119 (c.1437), ff. 141?
- Florence, Ashb.608 (c.1440) ff. 101v-106r
- London, Add.8784 (1442) ff. 138v-140v
- London, Add.10363 (c.1444) ff. 149r-152r
Abbaco remainder tables

London, Add.10363 (c.1444) ff. 149r-152r

• From 8 to 113
• Remainders for moduli 3, 5, 7
• Looking for remainders 2, 3, 1
Part of a number of remainder problems

- Volendo dire truovami 11 numeri tali che ciascuno de detti numeri sia partito per 3 e rimangha 2 e ssia partito per 5 rimangha 3 e sia partito per 7 rimangha j°. [f.150r]

- Sappi si chome puoi avere inteso lo primo numero sie 8 e llo secondo numero sie 113 pero che nnoi giugniamo sopra 8, 113 el 1/3 numero sie 218 el ¼ numero sie 323 el 1/5 numero sie 428 el 1/6 numero sie 533 el 1/7 numero sie 638 lo 1/8 numero sie 743 el 1/9 numero sie 848 el 1/10 numero sie 953 el 1/11 numero sie 1058.

No explanation which reveals a method such as the Ta-yen rule or Fibonacci-like method
Abbaco remainder tables

generalization by mathematical induction

• Ancora doverremo sappe sicome detto abbiamo se partira 8 per 3 e per 5 e per 7 si tti rimarra 2, 3, 1 e ssimigliantemente se partirai 113 per 3 per 5 per 7 si tti rimarra 2 e 3 e j° e ancora se pparti 9 per lo decto modo si nne viene tanto quanto di 114 e pero ch‘io giungho sopra 8 j° fa 9 e sopra 113 j° fa 114 e ssimigliantemente tanto vienne di 10 quanto di 115 e ttanto 11 quanto di 116 e ttanto di 12 quanto 117 et cosi ne verrebbe tanto dell’uno numero quanto dell’altro seguendo da 8 per insino in 13 e da 113 per insino in 218 chesciendo di ciascuna parte j° cioe dire 219 e 314 tucta via partendo per 3 per 5 e per 7 e questa reghola non arebbe mai fine e chiamasi la reghola del numero sanza fine.
Abbaco tradition: conclusion

Remainder tables have three functions:

• Finding the remainders by exhaustive method
• Finding 2 solutions by trial and error
• Acts as a ‘proof’ for the infinity of solutions

Importance:

• First (and only?) occurrence of proof by mathematical induction in abbaco texts
• First (and only?) reference to infinity
Conclusion

• Abbaco tradition is a mathematical culture ‘par excellence’
• Prime example of formal enculturation in mathematical culture
• Exerted an important impact on the mercantile society during the 14\textsuperscript{th}-15\textsuperscript{th} Cent.
• Paved the road for the transformation of mathematics in the 16\textsuperscript{th} century
  – Beginning of symbolism
  – Abstraction and expansion of the number concept
  – Influence on Humanist mathematics
Thank you